New BPS algebras from superstring compactifications

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Based mostly on

- hep-th/2107.03507 (w. Paquette, Persson and Volpato)

And also

- hep-th/1803.10798 (w. Paquette and Volpato)
- hep-th/2009.14710 (w. Paquette, Persson and Volpato)

Natalie Paquette

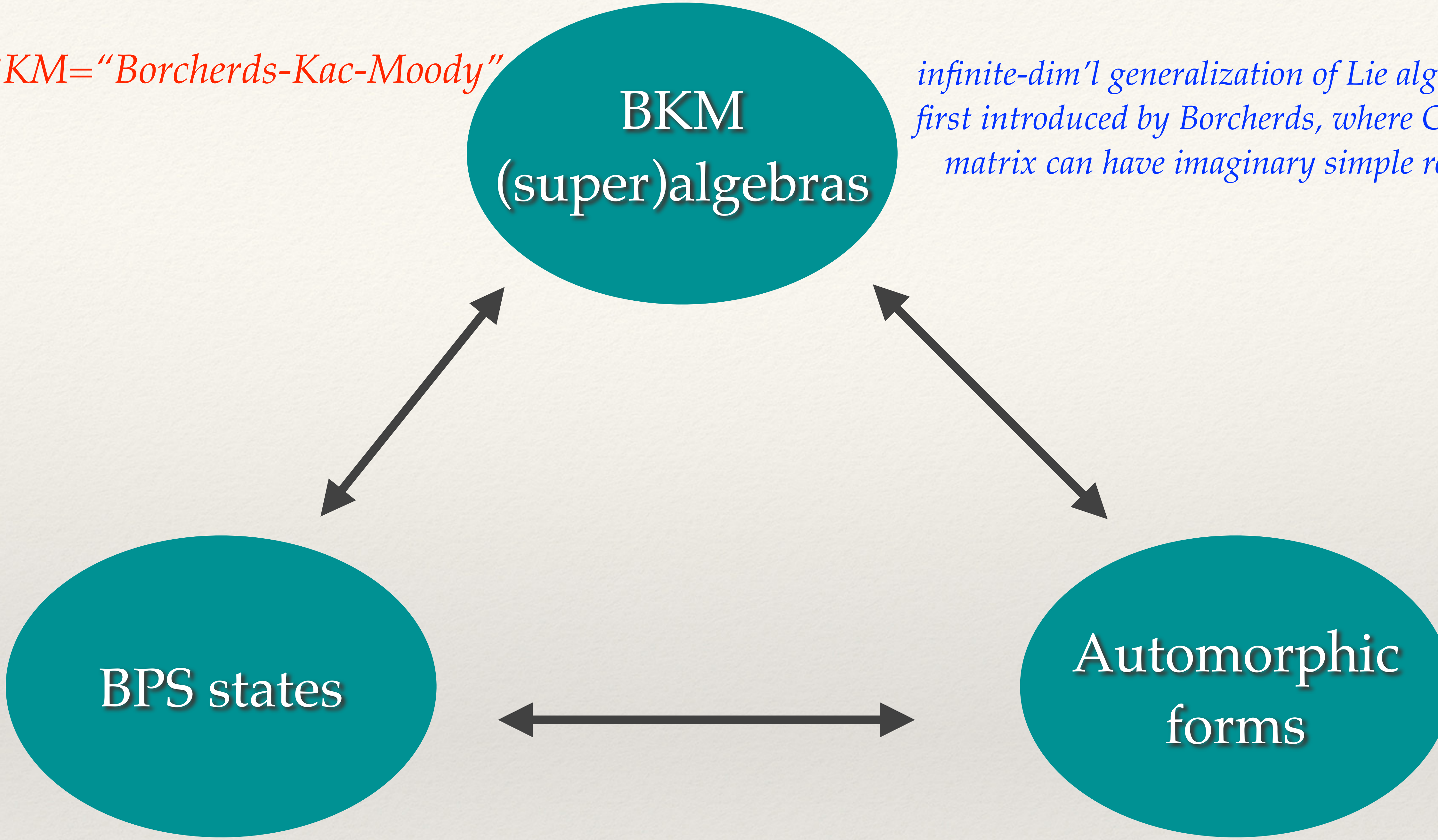
Daniel Persson

Roberto Volpato

See also: talk by N. Paquette at this series from October 2020

BKM="Borcherds-Kac-Moody"

infinite-dim'l generalization of Lie algebras, first introduced by Borcherds, where Cartan matrix can have imaginary simple roots



Broad Motivation

What are the symmetries underlying the structure of string theory?

More particularly, so-called BPS have played in important role in the physics of quantum field theories and string dualities and have been observed to have, e.g., interesting algebraic structure and connections to geometry

Moonshine—connection between finite groups and modular forms:
what is the role of the sporadic groups in physics & can we understand the beautiful properties inherent in moonshine physically?

What are "BKM algebras"?

Begin with this famous formula satisfied by the J function:

$$J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}$$

unique modular function for $SL(2,\mathbb{Z})$ with such Fourier exp. as $\tau \to i\infty$

where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$ and

$$J(\tau) = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + \underline{196884} q + \ldots$$

$= 1 + 196883$ *(dim. of smallest irrep. of $\mathbb{M}$, monster sporadic simple group)*

What are "Borcherds algebras"?

$$\boxed{J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}}$$

Borcherds: this is the denominator formula for an infinite-dimensional Lie algebra $\mathfrak{m}$

(c.f. Weyl denominator formula for Lie algebras:

$$\sum_{w\in W} \epsilon(w) e^{w(\rho)} = \prod_{\alpha\in\Delta^+} (e^{\alpha/2} - e^{-\alpha/2})\ )$$

This algebra is an example of a Borcherds-Kac-Moody (BKM) algebra—infinite-dimensional generalizations Lie algebras which can have imaginary simple roots α; i.e. such that $\langle \alpha, \alpha \rangle < 0$

This algebra has a natural action of the monster group $\mathbb{M}$

Monstrous moonshine

Borcherds' motivation was his proof of the monstrous moonshine conjectures of Conway and Norton:

There exists $\mathbb{Z}$-graded vector space $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$

such that $V_n^\natural$ is a finite-dim'l rep of $\mathbb{M}$ and

$$J(\tau) = \mathrm{Tr}_{V_n^\natural}\, q^n$$

Further, $\forall g \in \mathbb{M}$, the McKay-Thompson series:

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{Tr}_{V_n^\natural}\, g q^n$$

is a hauptmodul for a genus zero group $\Gamma_g < SL(2,\mathbb{R})$

biholomorphic map from $\mathbb{H}/\Gamma \to$ Riemann sphere

$\mathbb{H}/\Gamma$ has topology of Riemann sphere

Monstrous moonshine

A construction of $V^\natural$ was furnished by Frenkel-Lepowsky-Meurmann as a chiral vertex operator algebra (VOA) of $c = 24$ from a $\mathbb{Z}_2$ orbifold of chiral bosons on $\mathbb{R}^{24}/\Lambda_{Leech}$

Borcherds' proof involves constructing Lie algebra of physical states based on string-theory-inspired BRST reduction of vertex algebra

$$V\natural \otimes V^{\mathbb{R}^2/\Gamma^{1,1}} \otimes V^{ghost}$$

and considering "twisted denominator identities"

$$T_g(\sigma) - T_g(\tau) \sim p^{-1} \prod_{m,n} (1 - p^m q^n)^{\tilde{c}_g(mn)}$$

to prove hauptmodul property

Physics Motivation

Example 1: black holes from type II string theory on $K3 \times T^2$

$$d(P,Q) = \oint d\Omega \frac{e^{\pi i(\Omega,\Lambda)}}{\Phi_{10}(\Omega)}$$

$$\Lambda = (P,Q)$$

$$\Omega = \begin{pmatrix} \sigma & z \\ z & \tau \end{pmatrix}$$

degeneracy of "1/4-BPS" black holes with magnetic and electric charges P, Q

[Dijkgraaf, Verlinde, Verlinde]

Φ_{10} is the "denominator function" of a BKM superalgebra and a Siegel modular form of weight 10 for $Sp_2(\mathbb{Z})$

Proposal: this algebra governs physics of $1/4$-BPS black holes in $\mathcal{N} = 4$ string theory, including, e.g. degeneracies and decay processes

[e.g. Cheng, Verlinde]

Example 2: threshold corrections in $d = 4$, $\mathcal{N} = 2$ string theory (heterotic string on $K3 \times T^2$)

1-loop renormalization of gauge coupling (coming from BPS states) contains term

$$\frac{1}{g^2} \supset \log[J(iT) - J(iU)]$$

where T, U are moduli of the compactification

additive side of the denominator formula for $\mathfrak{m}$

Proposal: BPS states in string theory may form an algebra which is often a BKM algebra or generalization

[Harvey, Moore]

Monstrous BPS algebras

A physical reinterpretation of results of Borcherds: consider heterotic string theory on asymmetric orbifold of the form $T^8/\mathbb{Z}_2$, with worldsheet theory $V^\natural \otimes \overline{V^{f\natural}}$:

- similar BRST reduction shows space of physical BPS states in space time forms module over $\mathfrak{m}$
- genus zero groups Γ_g interpreted as T-duality groups in closely related "CHL models"

[Paquette, Persson, Volpato;
talk by N. Paquette]

Are there other contexts in string theory where we can construct explicit relations between Borcherds algebras, BPS states, and dualities, and can this give us insight into physical interpretations for other instances of moonshine?

Outline

1. New BKM superalgebras
2. Denominator formulas and simple roots
3. String theory models
4. Conclusions and speculation

BKM (super)algebras & chiral (S)VOAs

Beginning with chiral (S)VOA V of $c = 24$ $(c = 12)$, one can construct an associated BKM (super)algebra $\mathfrak{g}$ via a BRST reduction of chiral string theory with internal worldsheet theory V:

1. Consider $V^{tot} = V \otimes V_{\Gamma^{1,1}} \otimes V_{ghost}$
2. Perform GSO projection
3. Consider cohomology w.r.t nilpotent BRST supercharge Q, from which one can define Hilbert space of physical states $\mathscr{H}_{phys}$
4. $\mathscr{H}_{phys}$ has structure of a BKM (super)algebra $\mathfrak{g}$

to prove, one checks $\mathscr{H}_{phys}$ satisfies list of properties characterizing a BKM algebra

Note: in this construction, theories are chiral, and all spacetime dimensions are compactified!

BKM (super)algebras & chiral (S)VOAs

What V can we have?

$c = 24$ (bosonic case):

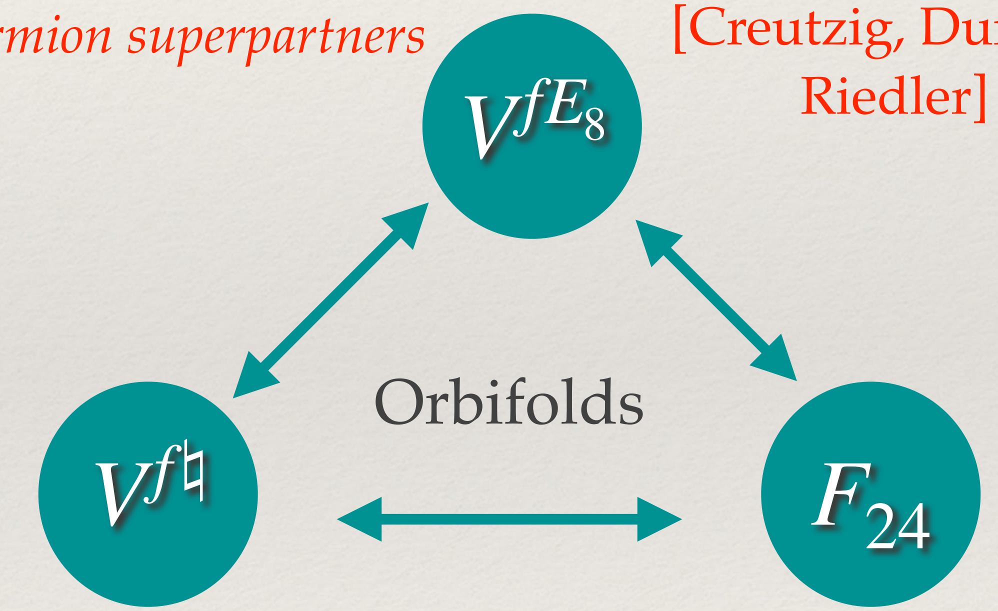
1. $V^\natural$, monster VOA
2. V_{Λ_N}, lattice VOA based on 24-dim'l Niemeier lattice Λ_N
3. V any other $c = 24$ holomorphic bosonic VOA (conjecturally 71 total) [Schellekens]

$c = 12$ (fermonic case):

3 unique self-dual SVOA of $c = 12$

chiral bosons on $\mathbb{R}^8/\Lambda_{E_8}$
+ 8 fermion superpartners

[Creutzig, Duncan, Riedler]



"Conway VOA"—unique theory with no fields of conformal weight 1/2

24 free chiral fermions

as SVOAs, they can be related to each other in numerous ways by gauging symmetries

Symmetries of these theories

$V^{f\natural}$ has a unique (up to isomorphism) choice of $\mathcal{N}=1$ supercurrent which is preserved by the sporadic simple group Co_1

$$\mathrm{sTr}_{V^{f\natural}}\, q^{L_0-c/24} = \frac{\eta^{24}(\tau/2)}{\eta^{24}(\tau)} + 24 = q^{-1/2} + 0 + 276q^{1/2} - 2048q + \ldots$$

=reps of Conway group

[Duncan]

Furthermore, there is a corresponding version of the genus zero property for the McKay-Thompson series of $V^{f\natural}$, such that we can think of it as the supersymmetric analogue of $V^\natural$

Is there a Conway (super)BKM and a physical explanation for the genus zero property of Conway moonshine?

F_{24} admits 8 distinct superconformal structures (up to iso.) of the form

$$G(z) \sim \sum_{ijk} c_{ijk} : \lambda_i \lambda_j \lambda_k :$$

where c_{ijk} are structure constants of semisimple Lie algebras g of dimension 24

The superconformal descendants of the 24 fermions are currents which generate an affine Kac-Moody algebra $\hat{g}$ based on g:

$$\boxed{\begin{gathered}(\widehat{su}(2)_2)^{\oplus 8}\ , \quad (\widehat{su}(3)_3)^{\oplus 3}\ , \quad \widehat{su}(4)_4 \oplus (\widehat{su}(2)_2)^{\oplus 3}\ , \quad \widehat{su}(5)_5\ , \quad \widehat{so}(5)_3 \oplus \hat{g}_{2,4}\ , \\ \widehat{so}(5)_3 \oplus \widehat{su}(3)_3 \oplus (\widehat{su}(2)_2)^{\oplus 2}\ , \quad \widehat{so}(7)_5 \oplus \widehat{su}(2)_2\ , \quad \widehat{sp}(6)_4 \oplus \widehat{su}(2)_2\ .\end{gathered}}$$

[Goddard and Olive]

each choice of superconformal structure will lead to a distinct BKM superalgebra which contains the corresponding affine Kac-Moody algebra as a sub algebra

There is an analogue of Borcherds construction and a corresponding super-BKM derived from worldsheet theories based on these $c = 12$ SVOAs:

1. V^{fE_8}: chiral superstring on $\mathbb{R}^8/\Lambda_{E_8}$ [Scheithauer]
2. $V^{f\natural}$: algebra of physical states in spacetime more directly related to $V^{s\natural}$ [SMH, Paquette, Volpato]
3. F_{24}: 8 distinct algebras based on choice of $\mathcal{N} = 1$ supercurrent [SMH, Paquette, Persson, Volpato]

we will forego discussing the construction of these algebras in detail here;
the method follows in a straightforward way from Borcherds' original work

Theorem: $\mathfrak{g}$ is a BKM superalgebra.

Proof: Consider V^{fE_8}, $V^{f\natural}$, F_{24} separately

[see Scheithauer; SMH , Paquette, Volpato; SMH, Paquette, Persson, Volpato, respectively]

Method: verify list of properties characterizing super-BKMs hold

Denominator formulas

For a BKM superalgebra of the form $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, sum of even and odd components, write $m_0(\alpha), m_1(\alpha)$ multiplicities of even/odd roots, we can write a (super)denominator formula, encoding info. about root spaces & real/imaginary simple roots:

$$e(-\rho)\sum_{w\in W}\det(w)w(T) = \frac{\prod_{\alpha\in\Delta_0^+}(1-e(-\alpha))^{m_0(\alpha)}}{\prod_{\alpha\in\Delta_1^+}(1+e(-\alpha))^{m_1(\alpha)}}$$

$$e(-\rho)\sum_{w\in W}\det(w)w(T') = \frac{\prod_{\alpha\in\Delta_0^+}(1-e(-\alpha))^{m_0(\alpha)}}{\prod_{\alpha\in\Delta_1^+}(1-e(-\alpha))^{m_1(\alpha)}}$$

where ρ is Weyl vector, Δ_0^+, Δ_1^+ are even/odd positive root sets, W is Weyl gp, and we define:

$$T := e^{-\rho}\sum_{\mu}(-1)^{ht(\mu)}e^{\mu},\ T' := e^{-\rho}\sum_{\mu}(-1)^{ht_0(\mu)}e^{\mu}$$

where $ht(\alpha) = \sum_{i\in\{\text{simples}\}} k_i,\ ht_0(\alpha) := \sum_{i\in\{\text{even simples}\}} k_i$

for roots $\alpha = \sum_{i\in\{\text{simples}\}} k_i\alpha_i$

Example 1: The Conway BKM

- rank 2 Cartan subalgebra
- roots: vectors $\alpha_{d,r} \in \Gamma^{1,1}$ where $(d, r) \in \mathbb{Z} \oplus \mathbb{Z}$, with $(\alpha, \alpha) = -2dr$
- positive roots: $\{(d, r) \,|\, d > 0\}$
- simple roots: $\{(1,r) \cup (d,0) \,|\, d > 0\}$

$$\frac{1}{p}\left[\prod_{d=1}^{\infty}\frac{1}{(1 \pm p^d)^{24}}\right]\frac{(1-pq)^{2048}(1-pq^2)^{49152}(1-p^2q)^{49152}\dots}{(1 \pm pq)^{2048}(1 \pm pq^2)^{49152}(1 \pm p^2q)^{49152}\dots}$$

$$= \frac{\eta^{24}(\sigma)}{\eta^{24}(2\sigma)} - 2^{12}\frac{\eta^{24}(2\tau)}{\eta^{24}(\tau)} \qquad \text{(denominator)}$$

$$= \frac{1}{\eta^{24}(\sigma)} \qquad \text{(super-denominator)}$$

Example 2: F_{24} with $\mathcal{N} = 1$ structure corresponding to A_1^8

- rank 10 Cartan subalgebra
- even roots: vectors $\alpha_{\vec{k}}$ where $\vec{k} = (m, n, w) \in \Gamma^{1,1} \oplus \Lambda_{A_1^8}$ such that $k^2 = -2mn + \langle w, w \rangle_{g=A_1^8}$ (& similar for odd)
- Weyl vector: $\rho = (-1, -1, \rho_{A_1^8})$
- real simple roots: vectors α_k such that $\langle \alpha_k, \alpha_k \rangle = 1$ and $\langle \alpha_k, \rho \rangle = 1/2$
- infinitely many real simple roots (all even, mult. 1)
- imaginary simple roots of the form $-n\rho$, $n \in \mathbb{N}$, all with mult. 8

Example 2: F_{24} with $\mathcal{N} = 1$ structure corresponding to A_1^8

Even roots: Fourier coefficients of the function

$$\frac{\eta(\tau)^8}{\eta(\tau/2)^8\eta(2\tau)^8} = 1/\sqrt{q} + 8 + 36\sqrt{q} + 128q + \dots$$

Odd roots: Fourier coefficients of the function

$$8\frac{\eta(\tau)^8}{\eta(2\tau)^{16}} = 8 + 128q + 1152q^2 + \dots$$

The additive side of the denominator formula:

$$\sum_{\mathrm{w}\in W} \det(\mathrm{w}) e^{-\mathrm{w}(\hat{\rho})} \prod_{n=1}^{\infty} (1 - e^{-n\mathrm{w}(\hat{\rho})})^{(-1)^n 8}$$

Full string theory constructions

We would like to explore the role of BKM (super)algebras in more "realistic" string theory constructions, i.e., ones where the worldsheet theory has both holomorphic and antiholomorphic sectors: $V \times \overline{W}$

There are many models we can consider where, where we take W to be one of the three $c = 12$ self-dual SVOA and V to be either

a) one of the same (type II) or

b) a holomorphic $c = 24$ bosonic VOA (heterotic)

The 2d models of interest

space-time massless spectrum

$$\{\mathcal{Q}^i, \mathcal{Q}^j\} = 2\delta^{ij}(P^0 - P^1)$$

Type II models

Theory	NS-NS	R-R	NS-R	R-NS	SUSY
$V^{fE_8} \times \bar{V}^{fE_8}$	8×8	8×8	8×8	8×8	$(16, 16)$
$V^{f\natural} \times \bar{V}^{fE_8}$	0	24×8	0	24×8	$(32, 8)$
$F_{24} \times \bar{V}^{fE_8}$	24×8	0	24×8	0	$(8, 8)$
$V^{f\natural} \times \bar{V}^{f\natural}$	0	24×24	0	0	$(24, 24)$ or $(48, 0)$
$F_{24} \times \bar{V}^{f\natural}$	0	0	24×24	0	$(24, 0)$
$F_{24} \times \bar{F}_{24}$	24×24	0	0	0	$(0, 0)$

Heterotic models

Theory	NS	R	SUSY
$V \times \bar{V}^{fE_8}$	$N \times 8$	$N \times 8$	$(8, 8)$
$V \times \bar{V}^{f\natural}$	0	$N \times 24$	$(24, 0)$
$V \times \bar{F}_{24}$	$N \times 24$	0	$(0, 0)$

V a holomorphic bosonic VOA with $c = 24$ and N currents

Most of these theories can arise as asymmetric orbifolds of type II/heterotic string on T^8 at holomorphically factorized point in moduli space

Physical BPS states

A similar BRST reduction to the chiral case—now combining holomorphic and antiholomorphic sectors—results in a Hilbert space of spacetime BPS states

$$\mathscr{H}_{BPS} := \left(\bigoplus_{\substack{k_l, k_r \neq 0 \\ k_l^0 = k_r^0 = k_r^1}} H^1(k_l) \otimes \bar{H}^1(k_r) \right) \oplus (\mathrm{H}^2(0,0))$$

cohomology at zero-mom.

left- and right-moving cohomology spaces (nonzero-mom.), arising from BRST reduction w.r.t
$Q = Q_L + Q_R$

Compactifying in spacetime on spacelike circle of radius *R*:

BPS condition: $k_r^0 = k_r^1$ (states annihilated by right-moving supercharge),

where $k_l^0 = k_r^0 = k_r^1 = \dfrac{1}{\sqrt{2}}(\dfrac{m}{R} - nR)$, $\qquad k_l^1 = \dfrac{1}{\sqrt{2}}(\dfrac{m}{R} + nR)$, $\qquad m, n \in \mathbb{Z}$

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cohomology at zero-mom.

left- and right-moving cohomology spaces (nonzero-mom.), arising from BRST reduction w.r.t
$Q = Q_L + Q_R$

We prove there is an action of a BKM algebra on the subspace of BPS states in this space of physical states, similar to what was proven in the case of $V^\natural \times \overline{V^{f\natural}}$

[SMH, Paquette, Persson, Volpato]

BKM algebras of physical states—the non-chiral case

For worldsheet theory $V_L \otimes \overline{V}_R$, $\exists$ algebras $\mathfrak{g}$ associated to V_L and $\bar{\mathfrak{g}}$ associated to $\overline{V}_R$, s.t.:

- $\mathfrak{g} = \oplus_{k\in\Gamma^{1,1}} \mathfrak{g}(k) = \oplus_{m,n\in\mathbb{Z}} \mathfrak{g}(m,n)$
- $\mathfrak{g}(k) \cong H^1(k) \ , \qquad\qquad k \in \Gamma^{1,1}$
- similar relations for $\bar{\mathfrak{g}}$

At nonzero mom., BPS states are holomorphically factorized of the form

$u \otimes \bar{v} \in \mathscr{H}_{BPS}$, where $u \in H^1(k_l)$ and $\bar{v} \in \overline{H}^1(k_r)$

Roughly, right-moving part of BPS states furnishes N copies of trivial rep. of $\bar{\mathfrak{g}}$, where N is number of right-moving supercharges, whereas left-moving part furnishes adjoint rep of $\mathfrak{g}$

BKM algebras of physical states—the non-chiral case

For $x \in \mathfrak{g}$, one can construct a representation δ_x which acts on states of the form $u \otimes \bar{v} \in \mathscr{H}_{\text{BPS}}$, where explicitly

$$\delta_x(u \otimes \bar{v}) := [x, u] \otimes (e^{\frac{i}{\sqrt{2}}(\frac{m}{R} - nR)X_r})_0 \bar{v}$$

These x are in fact associated with BRST exact states living in right-moving NS sector and generate algebra action on space of physical states $\mathscr{H}_{BPS}$

There exist subtleties for states at zero momentum, but one can prove something similar with a little more work

Furthermore, one can show:

$$\delta_x\left(\int_{\mathscr{M}_{0,n}} \langle \prod_{i=1}^{n} V_{u_i}(z_i, \bar{z}_i)\rangle\right) \equiv \sum_{j=1}^{n} \int_{\mathscr{M}_{0,n}} \langle V_{\delta_x(u_j)} \prod_{i \neq j} V_{u_i}\rangle = 0$$

genus 0, n-point correlator of BPS vertex operators

$\mathfrak{g}$ is symmetry of BPS amplitudes

Spacetime indices and denominator formulas

We can make connections to further properties of BKM algebras computing a supersymmetric index in the spacetime theory on a spatial circle of radius R and Euclidean time corresponding to inverse temperature β:

Define an index

$$Z = \mathrm{Tr}(-1)^F e^{-\beta H} e^{2\pi i w W} e^{2\pi i m M} \prod_i y_i^{q_i},$$

where H, W, M are the Hamiltonian, winding, and momentum, $(-1)^F$ is spacetime fermion number, and q_i are charges w.r.t. spacetime gauge gp

$$H = \frac{1}{\sqrt{2}} \left(\frac{M}{R} + WR \right)$$

$y_i = e^{2\pi i A_i}$, *A_i Wilson lines for spacetime gauge group along circle*

Receives contributions only from single string BPS states

Spacetime indices and denominator formulas

It is convenient to define complex parameters

$$T = w + i\frac{\beta R}{2\sqrt{2}\pi} \text{ and } U = m + i\frac{\beta}{2\sqrt{2}\pi R}, \, T, U \in \mathbb{H}$$

which parametrize Kahler and complex structure of spacetime T^2

This allows us to write

$$Z(T, U, A_k) = \mathrm{Tr}_{\mathrm{BPS}}(e^{2\pi i T W} e^{2\pi i U M} \prod_i e^{2\pi i \sum_k A_k q_k}(-1)^F) = \mathrm{Tr}_{\mathrm{BPS}}(p^W q^M \prod_k y_k^{q_k}(-1)^F)$$

where $\;p := e^{2\pi i T}\;,\;\; q := e^{2\pi i U}$

This index is a signed count spacetime BPS states, and should be closely related to the denominator or superdenominator function of the corresponding BKM algebra $\mathfrak{g}$ associated to V_L

Example 1: Consider $V_L \times \overline{V}_R = V_{\Lambda_N} \times V^{f\natural}$, where V_{Λ_N} is holomorphic, $c = 24$ VOA constructed from $\mathbb{R}^{24}/\Lambda_N$, where Λ_N is Leech lattice or one of 23 Niemeier lattices

(Heterotic)

$$Z \sim p^{24w_0} q^{24m_0} y^{-24\rho} \prod_{\substack{m, w \in \mathbb{Z}, \ell \in \mathbb{Z}^{24} \\ (m, w, \ell) > 0}} \left(1 - p^w q^m \prod_i y_i^{\ell_i}\right)^{24c(mw, \ell)}$$

$$\text{where } \mathrm{Tr}_{V_{\Lambda_N}} \, q^{L_0 - c/24} \prod_i y_i^{q_i} = \frac{\Theta_{\Lambda_N}(\tau, \xi)}{\eta(\tau)^{24}} = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^{24}} c(n, \ell) q^n \prod_i y_i^{q_i}$$

Z is the (24th power of the) product side of a denominator for the BKM based on V_{Λ_N}

Something similar is true in most of our models

Example 2: Consider $V_L \times \overline{V}_R = V^{f\natural} \times \overline{V}^{f\natural}$ (Type II)

$$Z \sim \left(q^{-1}\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^{24}}\right)^{24} = \left(\frac{1}{\eta^{24}(T)}\right)^{24}$$

OR

$$Z \sim \left(p^{-1}\prod_{w=1}^{\infty}\frac{1}{(1-p^w)^{24}}\right)^{24} = \left(\frac{1}{\eta^{24}(U)}\right)^{24}$$

depending on whether we assign +/- sign to 24 Ramond ground states of weight 1/2 in $V^{f\natural}$, in + case, space-time BPS states only carry momentum (& no winding) and in - only winding (no mom.)

$\boxed{\text{$Z$ is the (24th power of the) product side of the superdenominator for the Conway BKM}}$

And computing a "modified index" with $(-1)^{F_R}$ instead of $(-1)^F$ yields (24th power of) denominator formula of Conway BKM:

$$Z \sim \left(q^{-1}\prod_{m=1}^{\infty}\frac{1}{(1+q^m)^{24}}\prod_{w=1}^{\infty}\frac{(1-p^w q^m)^{c_{NS}(mw)}}{(1+p^w q^m)^{c_R(mw)}}\right)^{24}$$

OR

$$Z \sim \left(p^{-1}\prod_{w=1}^{\infty}\frac{1}{(1+p^w)^{24}}\prod_{m=1}^{\infty}\frac{(1-p^w q^m)^{c_{NS}(mw)}}{(1+p^w q^m)^{c_R(mw)}}\right)^{24}$$

Path integrals & theta lifts

One can compute second-quantized partition function $\mathcal{Z}$ via Euclidean path integral on a 2d space-time torus with Kahler and complex structure moduli T, U:

$$\mathcal{Z} \sim \exp\left(\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \frac{d^2\tau}{\tau_2^2} \mathrm{Tr}(\mathsf{q}^{L_0}\bar{\mathsf{q}}^{\bar{L}_0}(-1)^F)\right)$$

τ= complex structure of worldsheet, $\mathsf{q} = e^{2\pi i\tau}$

Exponent in $\mathcal{Z}$ reduces to computing integral of form:

$$\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\tau)\Theta_{\Gamma^{2,2}}(\tau; T, U)\frac{d^2\tau}{\tau_2^2}$$

Partition function for internal worldsheet CFT $V_L \otimes \overline{V}_R$

Siegel-Narain theta function for even unimodular lattice $\Gamma^{2,2}$ capturing momenta and winding along spacetime T^2 (no Wilson lines)

Path integrals & theta lifts

This kind of integral is called a ***theta lift*** in number theory

One starts with a modular form f on $\mathbb{H}$ and "lifts" it to an automorphic form $\Theta_f(g)$ on

$$\Gamma \backslash SO(m,n;\mathbb{R})/(SO(m)\times SO(n))$$

via the map

$$f \quad \longmapsto \quad \Theta_f(g) = \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\tau)\Theta_{\Gamma^{m,n}}(\tau;g)\frac{d^2\tau}{\tau_2^2}$$

Siegel-Narain theta function for lattice $\Gamma^{m,n}$, such more general lattices can arise if we include Wilson lines along spacetime torus

Captures connection between *automorphic forms*, *BKM algebras*, and *spacetime BPS states*

Path integrals & theta lifts

$\mathcal{Z}$ agrees with (super)denominator arising from BPS index Z*

Example: Type II on $V^{f\natural} \otimes \overline{V}^{f\natural}$

$$\mathcal{Z} \sim ||\eta^{-24}(T)||^{48}_{\mathrm{Pet}} \;\; \text{OR} \;\; \mathcal{Z} \sim ||\eta^{-24}(U)||^{48}_{\mathrm{Pet}}$$

Holomorphic half of $\mathcal{Z}$ reproduces supersymmetric index Z, the (24th power of) Conway BKM superdenominator

**More precisely, since the theory contains massless chiral particles, one must take only the holomorphic piece of $\mathcal{Z}$ to compare with Z*

Conclusions & speculation

We've constructed new examples of super-BKM algebras, and sketched how they may arise in string compactifications to 2d where the worldsheet theory has the form $V_L \times \overline{V}_R$

We show that in these theories

1. Spacetime BPS states furnish a representation of $\mathfrak{g}$
2. $\mathfrak{g}$ acts as a symmetry of certain BPS amplitudes
3. A suitably-defined spacetime supersymmetric index (as a trace in Hilbert space of physical states) reproduces denominator formula for $\mathfrak{g}$ (and is an automorphic form)
4. A path integral formulation of this index reduces to familiar "theta lift" from number theory

SMH, Paquette, Persson, Volpato

Conclusions & speculation

We hope such systems will be of use for exploring and understanding the role of BKM algebras and/or BPS algebras in string theory

We expect these different algebras may have fascinating/surprising relations among each other in the full physical string theory setting by considering the action of dualities (descending from their close relations at the SVOA level)

CHL orbifolds

Studying CHL orbifolds of $V^\natural \times \overline{V}^{f\natural}$ allows one to make connection to monstrous moonshine and "twisted denominator identities" & furnishes physical explanation of genus zero property of monstrous moonshine

What can we learn from CHL orbifolds of these more general models? In particular, can we understand the genus zero property of Conway moonshine in a similar way by considering CHL orbifolds of $V^{f\natural} \times \overline{V}^{f\natural}$?

BKM algebras in higher dimensions

How do BKM algebras in our models behave under decompactification?

Can we find similar instances of BKM/BPS algebras of this type in higher-dimensional string compactifications?

How does this connect with automorphic forms, geometry, and observations about threshold corrections & black holes mentioned in introduction?

Automorphic forms & dualities

The denominator function for the A_1^8 *BKM algebra (from choice of current algebra in* F_{24}*) arises from an expansion of an automorphic form* Ψ *on*

$\Gamma \backslash SO(2,10)/SO(2)\times SO(10)$ ← *closely related to moduli space of "Enriques Calami-Yau threefold"*

at a "level 2 cusp"

If you expand Ψ *at a "level 1 cusp", you get the denominator function for Scheithauer's super-BKM based on* V^{fE_8}

These expansions are related by transformations in the discrete group Γ*, which in string theory has the interpretation of a duality transformation between theory on* $F_{24}\times\overline{V}^{f\natural}$ *and* $V^{fE_8}\times\overline{V}^{f\natural}$

A similar phenomenon occurs for the Conway BKM and an automorphic form for $\mathscr{H}\times\mathscr{H}/\Gamma$*,* $\Gamma < O(2,2,\mathbb{Z})$ *: there is a second cusp where Fourier expansion leads to a denominator formula for a distinct super-BKM arising on a CHL orbifold of the theory*

Physically, the cusps represent different perturbative duality frames, where the states in the Fourier expansion of the denominator formula at one cusp are perturbative BPS states, where the states associated with the other cusp are non-perturbative

Moonshine & K3 string theory?

Some intriguing observations:

- $V^{s\natural}_{tw}$ (VOA closely related to $V^{f\natural}$) has a relation to elliptic genus of non-linear sigma models on K3 due to a construction of Duncan and Mack-Crane:

$$V^{s\natural}_{tw} \supset \widehat{su}(2)_1 \text{ such that } \mathrm{Tr}_{V^{s\natural}_{tw}}(-1)^F q^{L_0 - c/24} y^{2J_3} = \mathrm{EG}(K3)$$

and produces many McKay-Thompson series of Mathieu moonshine

- The superdenominator of the Conway BKM $\sim \dfrac{1}{\eta^{24}}$, generating fn of Euler characteristics of $Sym^N(K3)$, also spectrum of 1/2-BPS (Dabholkar-Harvey) states in $\mathcal{N} = 4$ string theory on $K3 \times T^2$

Is there a duality frame in which the physical appearance of the Conway BKM algebra involves compactification on a K3 surface (perhaps an orbifold of $K3 \times T^4$)?

Is there a connection to Mathieu moonshine???

Thank you!